Shock formation and traffic jam induced by a crossing in the 1D asymmetric exclusion model

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Shock formation and traffic jam induced by a crossing in the 1D asymmetric exclusion model

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Abstract. The asymmetric simple-exclusion model on two-crossing one-dimensional lattices is considered for studying shock formation and traffic jam induced by a crossing. The condition for shock formation is derived for car densities p_1 and p_2 of the first and second street. The phase diagram and the dependence of the traffic current on the car densities are shown. We compare the present result with that of a car accident. Also, we investigate the shock formation induced by both a crossing and a car accident. It is found that the dynamical transitions occur successively with increasing car density. The phase diagram is shown.

1. Introduction

The one-dimensional (1D) exclusion model is one of the simplest examples of a driven system [1, 2]. The model has been extensively studied for understanding systems of interacting particles [3, 4]. The 1D exclusion models are used to study the microscopic structure of shocks [5, 6] and are closely linked to growth processes [7–9]. The 1D asymmetric simple-exclusion model can be formulated as traffic jam problems.

Recently, traffic problem have attracted considerable attention. Traffic simulations based on various hydrodynamic models have provided much insight [10, 11]. However, the simulation of traffic flow is a formidable task since it involves many degrees of freedom. Cellular automaton (CA) models are being applied successfully to simulations of complex physical systems [12, 13]. Very recently, Biham *et al* [14] applied the two-dimensional asymmetric exclusion model to the traffic-jam problem. They found that a dynamical jamming transition between the low-density moving phase and the high-density jamming phase occurs at a critical density of cars. Also, in order to simulate freeway traffic, Nagel and Schreckenberg [15] extended the 1D asymmetric exclusion model to take into account car velocity. They showed that a transition from laminar traffic flow to start–stop waves occurs with increasing car density as is observed in real freeway traffic.

In real traffic-flow systems, the traffic jam is frequently induced by a single crossing when a street (first street) crosses with another street (second street). The congested street prevents cars from crossing its street. As soon as a single street begins to be congested, a traffic jam spreads from the crossing throughout another street. The occurence of a traffic jam on the first street strongly depends on whether or not the second street (crossing with the first street) becomes congested. The traffic jam appears as a shock (a discontinuity of densities) which separates between the low-density and the high-density traffic flow. The mechanism of shock formation in the 1D asymmetric simple-exclusion model was examined by Janowsky and Lebowitz [5]. They studied a shock structure when the translation invariance is broken by the insertion of a blockage. The model corresponds to the traffic-jam problem induced by a car accident. The

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mechanism of shock formation induced by the car accident will be different from that induced by the crossing. However, the shock formation and the traffic jam induced by a crossing have not been studied until now.

In this paper, we present the extended verision of the 1D asymmetric simple-exclusion model in which two one-dimensional lattices cross one another. We investigate the shock formation and traffic jam induced by a crossing. We find the phase diagram for shock formation against the car densities. We show the dependence of traffic current on the car densities. We compare our result with that of Janowsky and Lebowitz [5]. We find that the shock formation induced by a crossing is different from that of a car accident. Furthermore, we study the combined effect of both crossing and car accident on shock formation and traffic jam.

The organization of the paper is as follows. In section 2, we present our model. The traffic current and the phase diagram for shock formation are derived. The result is compared with that of Janowsky and Lebowitz. In section 3, we study the shock formation process induced by both crossing and car accident. We show the phase diagram for shock formation. Section 4 contains a brief summary.

2. Model and simulation result

Our model is defined on two one-dimensional lattices in which each lattice consists of L sites with periodic boundary conditions and the first lattice crosses with the second. Each site is either occupied by one car or is empty. Figure 1 shows the schematic representation of our model. The arrow pointing up (right) represents the car moving up (right). For an arbitrary configuration, one update of the system consists of the two steps. The first step is performed in parallel for all the cars on the first street (lattice) and the second step is performed in parallel for all the cars on the second street (lattice). The move or stop of cars in each step is the same as the 1D asymmetric simple-exclusion model. In each lattice, each arrow moves forward one step unless the forward nearest-neighbour site is occupied by another arrow. If an arrow is blocked ahead by another arrow, it does not move even if the blocking arrow moves out of the site during the same time step. In this model, the total number of cars on each street is conserved. The traffic problem on two streets connected by a crossing is reduced to its simplest form. The essential features are maintained. These features include the simultaneous flow of cars on two streets which cannot overlap and a car on the first (second) street at the crossing prevents cars on the second (first) street from going ahead. In the limit of no cars on the first or second street, our model reproduces the 1D asymmetric simple-exclusion model.

We have performed simulations of the CA model starting with an ensemble of random initial conditions where the system size $L = 2000$, and the initial densities of cars on the first and second streets are $p_1 = 0.0$–1.0 and $p_2 = 0.0$–1.0. Each run is calculated up to 10 000 time steps. The data are averaged over 50 runs. We present the simulation result obtained by the procedure explained above. We consider the traffic flow under the condition of a constant car density p_2 on the second street. Figure 2 shows the plot of the mean traffic-current J_1 on the first street against the car density p_1 on the first street for $p_2 \leqslant 0.5$ (circles), $p_2 = 0.65$ (triangles) and $p_2 = 0.8$ (squares). The mean current J is obtained by averaging over 5000 time steps except for the initial stage. For density p_2 lower than 0.5, the traffic flow of the second street does not affect the traffic current J_1 on the first street. The traffic current J_1 agrees with that of the 1D asymmetric exclusion model. Traffic flow on the first street is independent of that on the second street. For $p_2 = 0.65$, the traffic current J_1 becomes saturated at the car density $p_1 = 0.35 \pm 0.01$, remains at the constant value $J_1 = 0.35 \pm 0.01$ until $p_1 = 0.66 \pm 0.01$ and then decreases linearly with p_1. Similarly, for $p_2 = 0.8$, the traffic current J_1 becomes

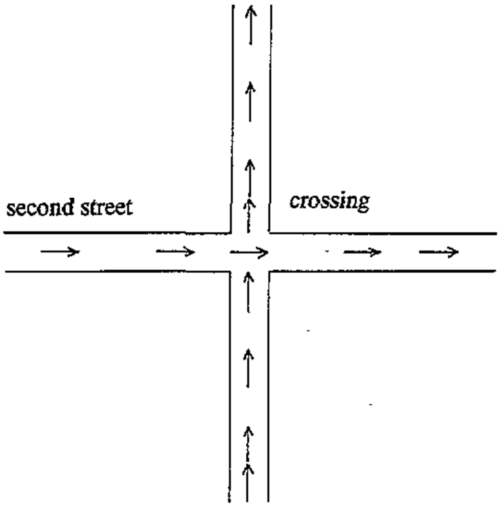

Figure 1. The schematic representation of traffic flow on two crossing streets. The arrow pointing up (right) represents the car moving up (right). A car on the first (second) street at the crossing prevents cars on the second (first) street from going ahead.

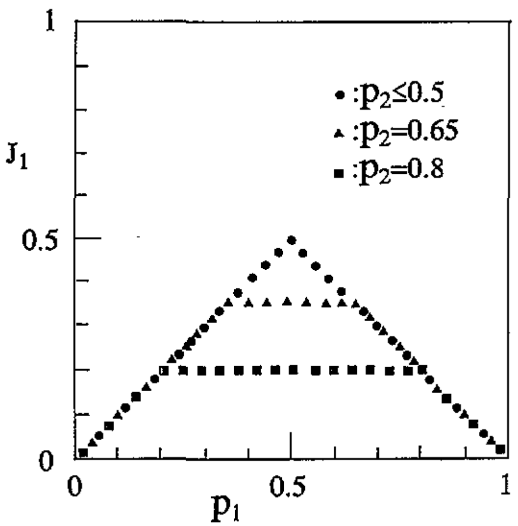

Figure 2. The plot of the mean traffic-current J_1 on the first street against the car density p_1 on the first street for $p_2 \leqslant 0.5$ (circles), $p_2 = 0.65$ (triangles) and $p_2 = 0.8$ (squares).

saturated at the density $p_1 = 0.2 \pm 0.01$, remains at the constant value $J_1 = 0.2 \pm 0.01$ until $p_1 = 0.81 \pm 0.01$ and then decreases linearly with p_1. The saturation of traffic current J_1 is due to the shock formation on the first street. The shock is a discontinuity of density which separates between the low density $p_{1,\text{low}}$ and the high density $p_{1,\text{high}}$. For illustration, figure 3 shows the typical configuration of cars up to 500 time steps for $p_1 = 0.5$ and $p_2 = 0.8$ where the system size $L = 190$. The upper and lower configurations represent respectively those on the second and first streets. The vertical direction indicates that of moving cars. The horizontal direction indicates that of time. A car is indicated by a dot. The trajectory of a car is represented by a line. The crossing is on the tops of the first and second streets. The shock (discontinuity of densities) is positioned near the centre of the first street. Figure 4 shows the density profiles $p_1(x)$ and $p_2(x)$ of the first and second streets against the distance x for the initial densities $p_1 = 0.5$ and $p_2 = 0.8$ where the system size $L = 2000$ and the density is averaged over 5000 time steps. The shock appears at $x = 1050$ on the first street. The shock separates between the low density $p_{1,\text{low}} = 0.2 \pm 0.01$ and the high density $p_{1,\text{high}} = 0.8 \pm 0.01$. The high density $p_{1,\text{high}}$ agrees with the density p_2 of the second street. When the shock formation occurs, the high density $p_{1,\text{high}}$ of the shock is determined by the density p_2. We find the following

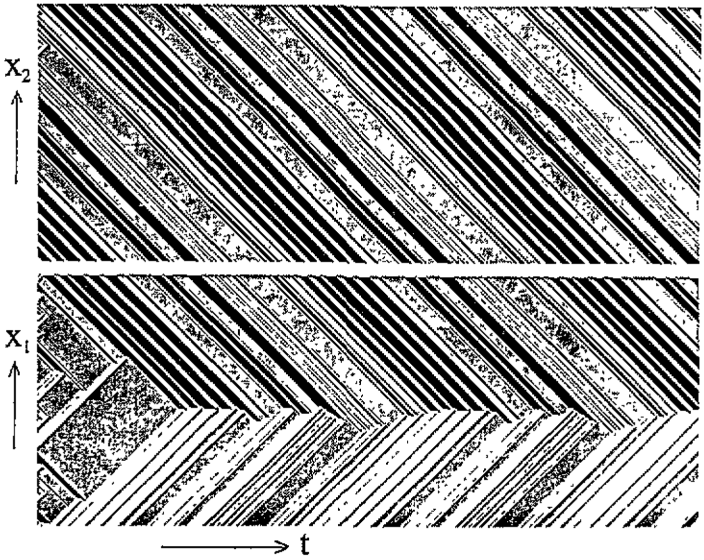

Figure 3. The typical configuration of cars up to 500 time steps for $p_1 = 0.5$ and $p_2 = 0.8$ where the system size $L = 190$. The upper and lower configurations represent, respectively, those on the second and first streets. The crossing is on the tops of the first and second streets. The upper direction indicates that of moving cars. The right direction indicates that of time. A car is indicated by a dot. The trajectory of a car is represented by a line. A shock (discontinuity of densities) appears near the centre of the first street.

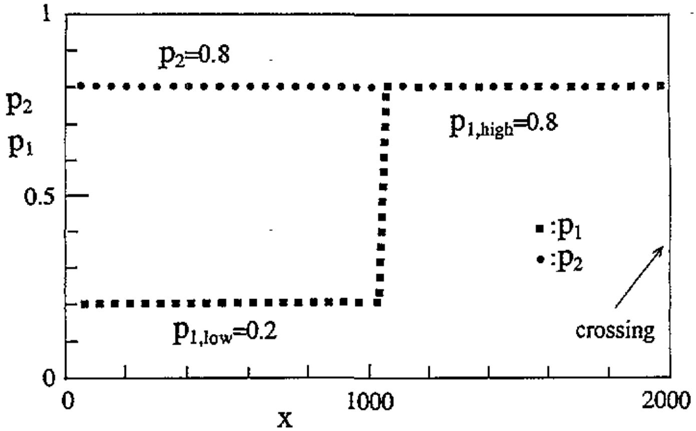

Figure 4. The density profiles $p_1(x)$ and $p_2(x)$ of the first and second streets against the distance x for the densities $p_1 = 0.5$ and $p_2 = 0.8$ where the system size $L = 2000$ and the density is averaged over 5000 time steps. The crossing is positioned at $x = 2000$. The shock appears at $x = 1050$ on the first street. The shock separates between the low density $p_{1,\mathrm{low}} = 0.2 \pm 0.01$ and the high density $p_{1,\mathrm{high}} = 0.8 \pm 0.01$. The high density $p_{1,\mathrm{high}}$ agrees with the density p_2 of the second street.

relationships between the low and high densities:

$$p_{1,\mathrm{high}} = p_2 \qquad \text{and} \qquad p_{1,\mathrm{low}} + p_{1,\mathrm{high}} = 1\,. \tag{1}$$

The second equality can be derived from the conservation law of traffic current after and before

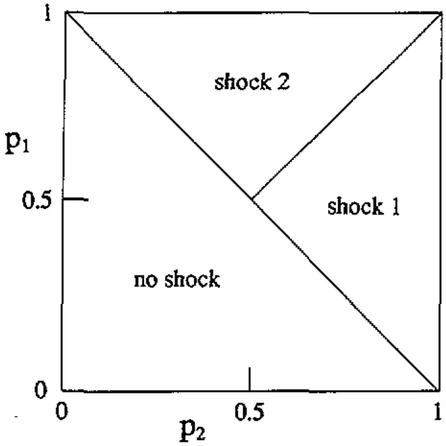

Figure 5. The phase diagram between p_1 and p_2. The region in which a shock appears on the first (second) street is indicated by 'shock 1' (shock 2).

the shock on the first street. The mean velocity $\langle v \rangle$ in the 1D asymmetric exclusion model is given by $\langle v \rangle = 1$ for $p \leqslant 0.5$ and $\langle v \rangle = (1-p)/p$ for $p > 0.5$. The conservation law of current after and before the shock is given by $p_{1,\mathrm{low}}\langle v_{1,\mathrm{low}} \rangle = p_{1,\mathrm{high}}\langle v_{1,\mathrm{high}} \rangle$ where $\langle v_{1,\mathrm{low}} \rangle$ and $\langle v_{1,\mathrm{high}} \rangle$ are the mean velocities before and after the shock. They are given by $\langle v_{1,\mathrm{low}} \rangle = 1$ and $\langle v_{1,\mathrm{high}} \rangle = (1 - p_{1,\mathrm{high}})/p_{1,\mathrm{high}}$. Therefore, the relationship (1) is satisfied. The shock appears when the density $p_2 \geqslant 0.5$. The shock front shifts backward with the initial density p_1. The position x_s/L of the shock is determined by the initial density p_1. It is given by

$$x_s/L = (p_{1,\mathrm{high}} - p_1)/(p_{1,\mathrm{high}} - p_{1,\mathrm{low}}) \qquad \text{for} \qquad p_1 \geqslant p_{1,\mathrm{low}} . \tag{2}$$

Figure 5 shows the phase diagram between p_1 and p_2. The shock does not appear under the following condition:

$$p_1 < 1 - p_2 . \tag{3}$$

A shock appears only on the first street under the condition

$$p_1 > 1 - p_2 \qquad \text{and} \qquad p_1 < p_2 . \tag{4}$$

The densities $p_{1,\mathrm{high}}$ and $p_{1,\mathrm{low}}$ in the front and back of the shock are given by (1). The shock position is given by (2). A shock appears only on the second street under the condition

$$p_1 > 1 - p_2 \qquad \text{and} \qquad p_1 > p_2 . \tag{5}$$

The densities $p_{2,\mathrm{high}}$ and $p_{2,\mathrm{low}}$ in the front and back of the shock are given by the same equation as (1) where the subscript 1 (2) is replaced by 2 (1). The shock position is also given by the same as (2) where the subscript 1 is replaced by 2. Shock appears on the street with lower car density. Shocks never appear simultaneously on both streets.

We compare our result with that of a car accident. Janowsky and Lebowitz investigated the shock formation induced by a car accident in the 1D asymmetric simple-exclusion model [5]. They showed the mechanism of shock formation when the translation invariance is broken by the insertion of a blockage: this reduces the rates at which particles jump across the bond by a factor r $(0 < r < 1)$. Then, shock formation occurs when the following condition is satisfied:

$$r/(r+1) < p < 1/(r+1) \tag{6}$$

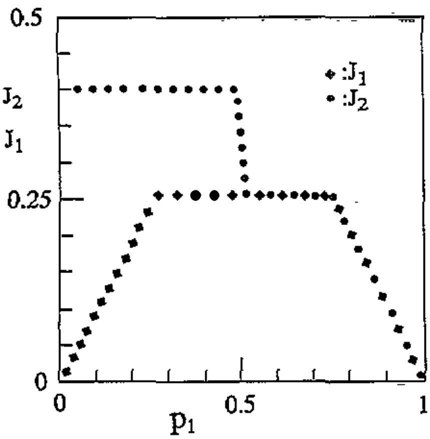

Figure 6. The plot of traffic currents J_1 (squares) and J_2 (circles) against the car density p_1 on the first street for $p_2 = 0.4$ and $r_1 = 0.35$.

where p is the car density and r is the rate representing the degree of blocking. The shock separates the low density p_{low} from the high density p_{high}. The densities p_{low} and p_{high} are given by

$$p_{\mathrm{low}} = r/(r+1) \qquad \text{and} \qquad p_{\mathrm{high}} = 1/(r+1)\,. \tag{7}$$

The condition (6) of shock formation induced by a blockage is definitely different from equation (4) and (5) of shock formation induced by a crossing. The densities p_{high} and p_{low} of (7) in the front and back of the shock are also different from (1) by a crossing.

Our model is fully deterministic since updating is performed in parallel. Even if updating is performed on a randomly chosen site, the results are the same.

3. Combined effect of both crossing and car accident

We consider shock formation and traffic jam induced by both the crossing and the car accident. We derive the phase diagram for shock formation and traffic jam. We assume that a car accident occurs at a position x_a on the first street. The distance x_a is measured from the position at which cars pass just the crossing: $x = 0$ and $x = L$ are positioned just after and just before the crossing. When a car flows through the car accident, it passes with probability r_1 or it stops with probability $1 - r_1$. The blockage of the car accident reduces the rate at which cars move across the position of car accident by a factor r_1 $(0 < r_1 < 1)$. We study the combined effect of both crossing and car accident on the shock formation and traffic jam. We put the blockage by car accident on the position $x_a = L/2$ on the first street. We calculate the traffic currents J_1 and J_2 on the first and second streets. Figure 6 shows the plot of traffic currents J_1 and J_2 against the car density p_1 on the first street for $p_2 = 0.4$ and $r_1 = 0.35$ where the system size $L = 2000$ and the currents are averaged over 5000 time steps. Cars on the first and second streets move at the maximum velocity $v_{\max} = 1.0$ until $p_1 = 0.26 \pm 0.02$. The traffic current J_1 is proportional to the car density p_1 on the first street. The traffic flow on the first street is not affected by that on the second street and the car accident on the first street. At density p_1 higher than 0.26, a shock induced by the car accident appears on the first street. It propagates from the car accident throughout space with increasing p_1. The traffic current J_1 on the first street becomes saturated and $J_1 = 0.26 \pm 0.02$ when the shock on the first street is formed. When p_1 becomes 0.5, the shock reaches the crossing. Then, the car density at the crossing becomes p_{high} (> 0.5). A new shock appears on the second street. Two shocks induced by the car accident and the crossing occur respectively on the first and second streets. The traffic current J_2 on the second street decreases sharply at $p_1 = 0.5$. Both traffic currents J_1 and J_2 become the constant value $J_1 = J_2 = 0.26 \pm 0.02$ until $p_1 = 0.74 \pm 0.02$. When the car density p_1 becomes larger than 0.74, the shock on the first street disappears. Only the shock on the second street appears. We find the following:

$$\begin{aligned}
&J_1 = p_1 \quad \text{and} \quad J_2 = 0.4 \quad \text{for} \quad 0 < p_1 < 0.26 \quad \text{(phase 1)} \\
&J_1 = 0.26 \quad \text{and} \quad J_2 = 0.4 \quad \text{for} \quad 0.26 < p_1 < 0.5 \quad \text{(phase 2)} \\
&J_1 = J_2 = 0.26 \quad \text{for} \quad 0.5 \leqslant p_1 < 0.74 \quad \text{(phase 3)} \\
&J - 1 = J_2 = 1 - p \quad \text{for} \quad 0.74 < p_1 < 1 \quad \text{(phase 5)}\,.
\end{aligned} \tag{8}$$

The dynamical phase transitions occur successively from the phase 1 (where the cars move with the maximum velocity), through phases 2 (the shock induced by the car accident appears on the first street) and 3 (two shocks induced by the car accident and the crossing appear respectively on the first and second streets), to phase 5 (a shock induced by the crossing appears only on the second street). Since the shock on the first street is induced by the car accident, the densities $p_{1,high}$ and $p_{1,low}$ in the front and back of shock on the first street are given by (7):

$$p_{1,high} = 1/(r_1 + 1) = 0.74 \quad \text{and} \quad p_{1,low} = r_1/(r_1 + 1) = 0.26 \tag{9}$$

where $r_1 = 0.35$. Since the shock on the second street is induced by the crossing, the densities $p_{2,high}$ and $p_{2,low}$ in the front and back of shock on the second street are given by (1):

$$\begin{aligned}
&p_{2,high} = p_{1,high} = 0.74 \quad \text{and} \quad p_{2,low} = 1 - p_{2,high} = 0.26 \\
&\qquad \text{for} \quad 0.5 \leqslant p_1 < 0.74 \\
&p_{2,high} = p_1 \quad \text{and} \quad p_{2,low} = 1 - p \quad \text{for} \quad 0.74 \leqslant -p_1 < 1\,.
\end{aligned} \tag{10}$$

For illustration, figure 7 shows the typical configurations of cars up to 500 time steps for $p_1 = 0.7$, $p_2 = 0.4$ and $r_1 = 0.35$ where $L = 190$. The upper and lower configurations represent respectively those on the second and first streets. The upper direction indicates that of moving cars. The fight direction indicates that of time. A car is indicated by a dot. The trajectory of a car is represented by a line. The crossing is on the tops of the first and second streets. The car accident is positioned at the centre of the first street. The shocks (discontinuities of densities) appear on the first and second streets.

We calculate traffic currents in the case of higher car density p_2. Figure 8 shows the plot of traffic currents J_1 and J_2 against the car density p_1 on the first street for $p_2 = 0.85$ and $r_1 = 0.35$ where the system size $L = 2000$ and the currents are averaged over 5000 time steps. Cars on the first and second streets move at the maximal velocity $v_{max} = 1.0$ until $p_1 = 0.15 \pm 0.02$. The traffic current J_1 is proportional to the car density p_1 on the first street.

The traffic flow on the first street is not affected by that on the second street and the car accident on the first street. At density p_1 higher than 0.15, a shock induced by the crossing appears on the first street. It propagates from the crossing throughout space with increasing p_1. The traffic current J_1 on the first street becomes saturated and equals J_2 ($J_1 = J_2 = 0.15 \pm 0.02$) when the shock on the first street is formed. In this case of higher density p_2, the shock on the first street is induced by the crossing. In the previous case of lower density p_2, the shock on

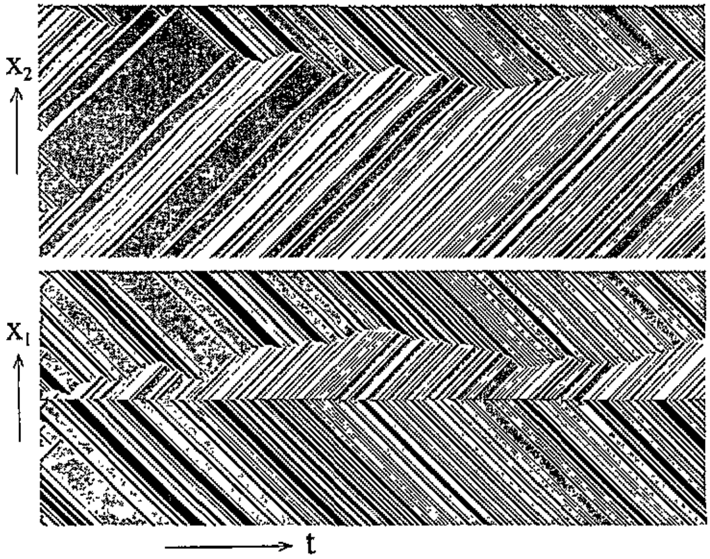

Figure 7. The typical configurations of cars up to 500 time steps for $p_1 = 0.7$, $p_2 = 0.4$ and $r_1 = 0.35$ where $L = 190$. The upper and lower configurations represent respectively those on the second and first streets. The upper direction indicates that of moving cars. The right direction indicates that of time. The crossing is on the tops of the first and second streets. The car accident is positioned at the centre of the first street. The shocks (discontinuities of densities) appear on the first and second streets.

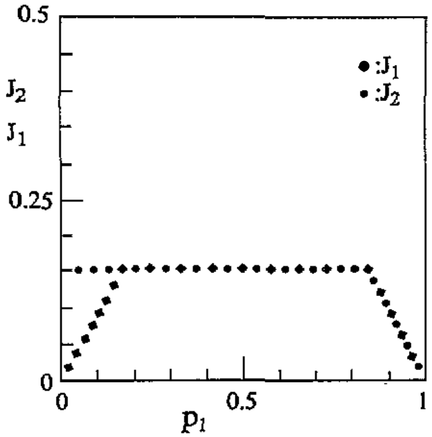

Figure 8. The plot of traffic currents J_1 (squares) and J_2 (circles) against the car density p_1 on the first street for $p_2 = 0.85$ and $r_1 = 0.35$.

the first street is induced by the car accident. Whether the shock on the first street is induced by the crossing or by the car accident depends on the car density p_2 on the second street. When the car density p_1 on the first street becomes larger than p_2, the shock on the first street disappears and a new shock on the second street is induced by the crossing. The current J_1 becomes equal to J_2 and decreases linearly with p_1. We find the following:

$$
\begin{aligned}
&J_1 = p_1 \quad \text{and} \quad J_2 = 0.15 \quad \text{for} \quad 0 < p < 0.15 \quad \text{(phase 1)}\\
&J_1 = J_2 = 0.15 \quad \text{for} \quad 0.15 \leqslant p_1 < 0.85 \quad \text{(phase 4)}\\
&J_1 = J_2 = 1 - p_1 \quad \text{for} \quad 0.85 \leqslant p_1 < 1 \quad \text{(phase 5)}.
\end{aligned}
\tag{11}
$$

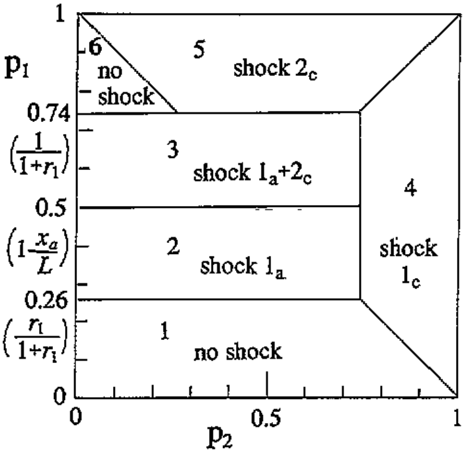

Figure 9. The phase diagram between p_1 and p_2. Six dynamical phases 1–6 appear. Each phase is indicated by number.

The dynamical phase transitions occur successively from phase 1 (where the cars move with the maximum velocity), through phase 4 (the shock induced by the crossing appears on the first street), to phase 5 (the shock induced by the crossing appears only on the second street). In this case of higher density p_2, the shock formation is not induced by the car accident but by the crossing. The dynamical phase transitions donot depend on the car accident. The transitions agree with those by the crossing explained in section 2. We calculate traffic flow for various values of p_2. We find the phase diagram for the shock formation process. Figure 9 shows the phase diagram between p_1 and p_2. Six dynamical phases 1–6 appear. Each phase is indicated by a number. The shock does not appear in phases 1 and 6. In phase 2, a shock induced by the car accident appears on the first street. In phase 3, a shock induced by the car accident appears on the first street and a shock induced by the crossing appears on the second street. In phase 4, a shock induced by the crossing appears on the first street. In phase 5, a shock induced by the crossing appears on the second street. The regions of each phase are divided by the following:

$$
\begin{array}{llll}
\text{phase 1 for} & 0 < p_1 \leqslant r_1/(1+r_1) & \text{and} & p_1 < 1 - p_2 \\
\text{phase 2 for} & r_1/(1+r_1) < p_1 < 1 - x_a/L & \text{and} & 0 < p_2 \leqslant 1 - r_1/(1+r_1) \\
\text{phase 3 for} & 1 - x_a, /L < p_1 < 1/(1+r_1) & \text{and} & 0 < p_2 \leqslant 1 - r_1/(1+r_1) \\
\text{phase 4 for} & 1 - p_2 < p_1 \leqslant p_2 & \text{and} & 1 - r_1/(1+r_1) < p_2 < 1 \\
\text{phase 5 for} & p_1 > 1 - p_2, p_1 > p_2 & \text{and} & p_1 > 1/(1+r_1) \\
\text{phase 6 for} & 1/(1+r_1) < p_1 \leqslant 1 - p_2 \,. & &
\end{array}
\tag{12}
$$

The densities before and after the shock induced by the crossing are obtained by (1). The densities before and after the shock induced by the car accident are given by (7).

4. Summary

We presented the extended version of the 1D asymmetric simple-exclusion model to investigate shock formation and traffic jam induced by a crossing. The extended model consists of two-crossing one-dimensional lattices. We calculated the traffic current on the system. We derived the condition of shock formation and showed the phase diagram. Also, we investigated the combined effect of crossing and car accident on shock formation. We showed that the dynamical phase transitions occur successively with increasing car density. We found the phase diagram.

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